Shelton, Anelia Sue. Fundamental Properties of Near-Rings. (1973)
Directed by: Dr. Robert L. Bernhardt. pp. 38

The basic properties of near-rings are examined, and are then used to develop several fundamental theorems, among which are the Factor Theorem, the Isomorphism Theorems, the Correspondence Theorem, and theorems concerning near-ring embeddings. Further, the ideals and modules of near-rings are characterized, and several theorems based on these concepts are proved. A number of examples of near-rings are included, with one of the examples being the set of functions from a group G into itself, denoted by $T(G)$. This near-ring, and one of its sub-near-rings, $T_0(G)$, are considered in detail, with a decomposition of $T_0(G)$ in terms of left ideals each isomorphic to G being proved.

FUNDAMENTAL PROPERTIES OF NEAR-RINGS

by

Anelia Sue Shelton

A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
March, 1973

Approved by

Robert L. Bernhardt
Thesis Advisor

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Advisor Robert L. Barnhardt

Oral Examination Committee Members Hughes B. Hoyle, III

E. E. Posey

William A. Powers III

April 3, 1973
Date of Examination

ACKNOWLEDGMENT

The author wishes to express her sincere appreciation to Dr. Robert L. Bernhardt for his patience and valuable assistance during the preparation of this thesis.

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INTRODUCTION

If G is an abelian group, it is well-known that the set of all group homomorphisms from G to G is a ring. If $T(G)$ denotes the set of all functions from G to G, however, then $T(G)$ is not a ring under addition and composition of functions; in particular, the left distributive law $f \circ (g + h) = f \circ g + f \circ h$ need not hold. It is perhaps because of examples similar to this that the concept of a near-ring was developed. It is the purpose of this thesis to study the elementary properties of near-rings, and to give several examples of near-rings.

Chapter I defines near-rings, near-ring ideals, and quotient near-rings. The major results in this chapter are the development of the Factor Theorem, the Isomorphism Theorems, and the Correspondence Theorem for near-rings.

In Chapter II, the transformation near-ring, $T(G)$, and several of its sub-near-rings are introduced and are used to develop fundamental theorems concerning near-ring embeddings. Among these results is the fact that any near-ring can be embedded in a near-ring with identity. It is further proved that isomorphisms and embeddings of groups implies isomorphisms and embeddings of associated near-rings.

Chapter III concentrates on a specific sub-near-ring, $T_0(G)$, of the transformation near-ring $T(G)$. When G is finite, it is proved that $T_0(G)$ can be decomposed into a direct sum of minimal left ideals, each of which is group isomorphic to G. Curiously, the theorem is definitely false when G is an infinite group. The minimal left ideals which are direct summands of $T_0(G)$ have a very accessible characterization; an ideal P_x consists of all functions from G to G which are 0 except at $x \in G$. Further, it is shown that P_x is generated by an idempotent function, which maps x onto x and maps the remainder of G onto 0.

CHAPTER I

ELEMENTARY PROPERTIES OF NEAR-RINGS

Definition 1.1. A (right) near-ring $(N,+,\cdot)$ is a set N together with two binary operations, addition, $+$, and multiplication, $\cdot$, on N such that:

(a) $(N,+)$ is a (not necessarily abelian) group;

(b) Multiplication is associative;

(c) $(a + b) c = ac + bc$ for all $a, b, c \in N$.

Usually the notation for the near-ring $(N,+,\cdot)$ is simplified with N being used to represent $(N,+,\cdot)$, and with N^+ being used to represent $(N,+)$. Also since we will be considering only right near-rings, it is understood that the terms "near-ring" and "sub-near-ring" will mean "right near-ring" and "right sub-near-ring".

It should be noted that some authors, such as C. J. Maxson [8], further include in the definition of a near-ring N, the following property:

(d) For every $a \in N$, $a0 = 0$, where 0 is the additive identity of N^+.

However, the inclusion of property (d) is not widely accepted, and its usage results in proofs for many of the following theorems becoming trivial. Hence we will not assume property (d).

Definition 1.2. If a near-ring N contains a non-zero multiplicative identity, then N is called a unitary near-ring.

Definition 1.3. A unitary near-ring in which every non-zero element has a multiplicative inverse is called a division near-ring. (Some authors call such a near-ring a near-field [8]).

Proposition 1.4. For every a and b in the near-ring N,

(a) $0a = 0$;

(b) $(-b)a = -(ba)$.

Proof: (a) Since $a + 0 = a$, it follows that $(a + 0)a = aa$. However, $(a + 0)a = aa + 0a$ which implies $aa + 0a = aa$. But $aa + 0 = aa$. Therefore $0a = 0$.

To show part (b), we start with the fact that $b + (-b) = 0$, from whence it follows that $[b + (-b)]\, a = 0a = 0$. But $[b + (-b)]\, a = ba + (-b)\, a$ which implies $ba + (-b)\, a = 0$. However, $ba + [-(ba)] = 0$. Thus $(-b)\, a = -(ba)$.

Remark 1.5. In a near-ring N, the following properties do not necessarily hold for $a, b \in N$:

(a) $a(-b) = -(ab)$ (see Example 1.8);

(b) $(-a)\,(-b) = ab$ (see Example 1.8).

Example 1.6. A natural example of a near-ring, which is not a ring, is the set $T(G)$ of mappings of a group G into itself, where mappings are added by adding images and multiplied by composition. We call $T(G)$ the transformation near-ring on G. We shall consider this example in greater detail in Chapters II and III.

Example 1.7. The following tables of multiplication and addition define a near-ring N containing the two elements 0, f:

Addition

+	0	f
0	0	f
f	f	0

Multiplication

·	0	f
0	0	0
f	f	f

Note that the left distributive property fails in several instances. Among these are the following:

$f[0 + f] = ff = f$ while $f0 + ff = f + f = 0$;

and $f[f + f] = f0 = f$ while $ff + ff = f + f = 0$.

Example 1.8. Let G be any additive group with identity 0. Define $xy = x$ for all $x, y \in G$. This definition of multiplication creates a near-ring in which commutativity of addition does not necessarily hold, and in which the left distributive property fails, since $z(x + y) = z$ while $zx + zy = z + z$ for $x, y, z \in G$.

Note that the right distributive property does hold, since

$$(x + y)z = x + y = xz + yz.$$

Also note that this near-ring provides counterexamples for Remark 1.5. To disprove (a), consider $a(-b)$ for $a, b \in G$. By definition of mutiplication on G, $a(-b) = a$ while $-(ab) = -a$. Part (b) can be seen to be false by noticing that for $a, b \in G$, $(-a)(-b) = -a$ while $ab = a$.

Example 1.9. The set of polynomials over a ring, with the operations of addition and composition, is a near-ring. In this near-ring, addition is commutative, but the left distributive property fails to hold since polynomials are not necessarily linear functions.

Example 1.10. The following tables of addition and multiplication define a unitary near-ring N, containing four elements 0, a, b, c:

Addition

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Multiplication

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	0	0
c	0	c	b	c

One instance of the left distributive property failing is $b(a + b) = bc = 0$ while $ba + bb = b + 0 = b$.

Definition 1.11. A near-ring homomorphism, or just homomorphism, is a mapping f of a near-ring N into a near-ring N' such that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)\, f(b)$ for every $a, b \in N$.

Definition 1.12. The set of all homomorphisms from the near-ring N into the near-ring N' is denoted by Hom (N,N').

Definition 1.13. An element f of Hom (N,N') is a near-ring isomorphism, or just isomorphism, provided there exists $g \in$ Hom (N',N) such that $g \circ f = 1_N$ and $f \circ g = 1_{N'}$.

Proposition 1.14. An element f of Hom (N,N') is an isomorphism if and only if f is one-to-one and onto.

<u>Proof</u>: ($\rightarrow$) Suppose an element f of Hom (N,N') is an isomorphism. Then there exists $g \in$ Hom (N',N) such that $g \circ f = 1_N$ and $f \circ g = 1_{N'}$. Suppose $f(a) = f(b)$ for $a, b \in N$. Then $g(f(a)) = g(f(b))$, from which it follows that $1_N(a) = 1_N(b)$. Hence $a = b$, which implies f is one-to-one. The function f is also onto, since for $d \in N'$, $g(d) \in N$. Thus $f(g(d)) = 1_{N'}(d) = d$. Therefore there exists $x \in N$ such that $f(x) = d$, namely $x = g(d)$.

($\leftarrow$) Suppose $f \in$ Hom (N,N') is one-to-one and onto. Let us define a function f^{-1} from N' into N. Suppose $n_1' \in N'$. Since f is one-to-one and onto, there exists $n_1 \in N$ such that $f(n_1) = n_1'$. Define $f^{-1}(n_1') = n_1$. Then f^{-1} is well defined since $n_1' = n_2'$ implies $f(n_1) = f(n_2)$, and because f is one-to-one, $n_1 = n_2$. To show f^{-1} is a homomorphism, we need to consider $f^{-1}(n_1' + n_2')$ and $f^{-1}(n_1' n_2')$ for $n_1', n_2' \in N'$. Since $n_1', n_2' \in N'$, $n_1' + n_2' = n_3'$ which is in N', and $n_1' n_2' = n_4'$ which is also in N'. Hence $f(n_1) + f(n_2) = f(n_3)$ or $f(n_1 + n_2) = f(n_3)$. But since f is one-to-one, this implies that $n_1 + n_2 = n_3$. Also $n_1' n_2' = n_4'$ leads to $f(n_1) f(n_2) = f(n_4)$. Therefore $f(n_1 n_2) = f(n_4)$ and $n_1 n_2 = n_4$. Hence

$$f^{-1}(n_1' + n_2') = f^{-1}(n_3') = n_3 = n_1 + n_2 = f^{-1}(n_1') + f^{-1}(n_2'),$$

and $f^{-1}(n_1' n_2') = f^{-1}(n_4') = n_4 = n_1 n_2 = f^{-1}(n_1') f^{-1}(n_2')$.

Therefore f^{-1} is a homomorphism from N' into N.

Moreover, for $n \in N$, $n' \in N'$,

$(f^{-1} \circ f)(n) = f^{-1}(f(n)) = f^{-1}(n') = n = 1_N(n)$,

and $(f \circ f^{-1})(n') = f(f^{-1}(n')) = f(n) = n' = 1_{N'}(n')$.

Therefore by Definition 1.13, f is an isomorphism.

Definition 1.15. A near-ring N is isomorphic to a near-ring N', denoted by $N \cong N'$, if there exists a near-ring isomorphism from N to N'.

Definition 1.16. Let N be a near-ring. A non-empty subset M of N is called a (right) sub-near-ring of N provided M, together with $+$ and $\cdot$ restricted to M, forms a near-ring.

Proposition 1.17. Let N be a near-ring and let $M \subseteq N$. Then M is a sub-near-ring of N if and only if $a - b \in M$ and $ab \in M$ for all $a, b \in M$.

Proof: ($\rightarrow$) Suppose M is a sub-near-ring of N. Then for $a, b \in M$, ab is obviously contained in M, and since $b \in M$, $-b$ is also in M. This implies that $a - b \in M$.

($\leftarrow$) Suppose $a - b \in M$ and $ab \in M$ for all $a, b \in M$. Then $b - b \in M$, from which it follows that $0 \in M$, and so $-b = 0 - b \in M$. Therefore M is a group under addition. Moreover, if $a, b, c \in M$, then $a, b, c \in N$, from whence it follows that $(ab)c = a(bc)$ and $(a + b)c = ac + bc$. Thus M is a sub-near-ring.

Proposition 1.18. Let N and N' be near-rings, and let f be an element of $\text{Hom}(N,N')$. Then $\text{Im } f$ is a sub-near-ring of N'. Also f is one-to-one if and only if $\text{Ker } f = \{0\}$.

Proof: Suppose N and N' are near-rings and $f \in \text{Hom}(N,N')$.

We shall first show that Im f is a sub-near-ring of N'. Let $x, y \in \text{Im } f$. Then there exists $a, b \in N$ such that $f(a) = x$ and $f(b) = y$. Since $a, b \in N$, $a - b$ and ab are also in N. This implies that $x - y = f(a) - f(b) = f(a - b) \in \text{Im } f$ and $xy = f(a)\, f(b) = f(ab) \in \text{Im } f$. Therefore Im f is a sub-near-ring of N'.

Now let us show that f is one-to-one if and only if $\text{Ker } f = \{0\}$. Suppose f is one-to-one and let $a \in \text{Ker } f$. Then $f(a) = 0 = f(0)$, from which it follows that $a = 0$. Hence $\text{Ker } f = \{0\}$.

Assume $\text{Ker } f = \{0\}$, and further suppose that $f(a) = f(b)$. Then $f(a) - f(b) = 0$, from which it follows that $f(a - b) = 0$. Thus $a - b \in \text{Ker } f$, so $a = b$. Hence f is one-to-one.

Definition 1.19. The two-sided ideals, or just ideals, of the near-ring N are defined to be the kernels of homomorphisms of N.

Theorem 1.20. (Blackett [3]): The two-sided ideals of the near-ring N are just the additive normal subgroups T of N such that:

(a) $TN \subseteq T$;

(b) $n_2\,(t + n_1) - n_2\, n_1 \in T$ for $n_1, n_2 \in N$ and $t \in T$.

Proof: ($\rightarrow$) Suppose T is a two-sided ideal of the near-ring N. Let $a \in TN$. Then $a = \sum_{i=1}^{k} t_i\, n_i$ where $t_i \in T$ and $n_i \in N$. But $T = \text{Ker } f$ where f is some homomorphism of N. Hence

$$f(a) = f\left(\sum_{i=1}^{k} t_i\, n_i\right) = \sum_{i=1}^{k} f(t_i\, n_i) = \sum_{i=1}^{k} f(t_i)\, f(n_i) = \sum_{i=1}^{k} 0 f(n_i) = 0.$$

Therefore $a \in \text{Ker } f = T$, so $TN \subseteq T$. Let $t \in T$ and $n_1, n_2 \in N$.

Then $f[n_2\ (t + n_1) - n_2\ n_1] = f\ [n_2\ (t + n_1)] - f(n_2\ n_1) =$

$f(n_2)\ f(t + n_1) - f(n_2)\ f(n_1) = f(n_2)\ [f(t) + f(n_1)] - f(n_2)\ f(n_1) =$

$f(n_2)\ f(n_1) - f(n_2)\ f(n_1) = 0.$

Thus $n_2\ (t + n_1) - n_2\ n_1 \in \text{Ker } f = T$. Finally, T is an additive normal subgroup of N because it is the kernel of a group homomorphism, namely f.

($\leftarrow$) Suppose T is any additive normal subgroup of the near-ring N such that $TN \subseteq T$ and $n_2\ (t + n_1) - n_2\ n_1 \in T$ for any $n_1, n_2 \in N$ and $t \in T$. Let π be the natural group homomorphism of N onto N/T defined by $\pi(n) = n + T$ for $n \in N$.

<u>Claim</u>: If multiplication on N/T is defined by $(n_1 + T)\ (n_2 + T) = n_1\ n_2 + T$, then N/T is a near-ring.

<u>Subproof</u>: Let $(n_1 + T)\ (n_2 + T)$ be defined on N/T as $n_1\ n_2 + T$. To show this multiplication is well-defined, assume $n_1 + T = n_2 + T$ and $n_3 + T = n_4 + T$ for $n_1, n_2, n_3, n_4 \in N$. Then $n_1 = n_2 + t_1$ and $n_3 = n_4 + t_2$ for $t_1, t_2 \in T$. Hence $n_1\ n_3 = (n_2 + t_1)\ (n_4 + t_2) = n_2\ (n_4 + t_2) + t_1\ (n_4 + t_2)$. Since $TN \subseteq T$ and T is an additive normal subgroup of N, $n_2\ (n_4 + t_2) + t_1\ (n_4 + t_2) = n_2\ (t_3 + n_4) + t_4$ for $t_3, t_4 \in T$. But $n_2\ (t_3 + n_4) - n_2\ n_4$ is contained in T, so $n_2\ (t_3 + n_4) = n_2\ n_4 + t_5$ for some $t_5 \in T$. Thus $n_1\ n_3 = n_2\ n_4 + t_6$ for $t_6 \in T$, which implies $n_1\ n_3 + T = n_2\ n_4 + T$. Hence multiplication is well defined.

Multiplication is also associative, since

$[(n_1 + T)(n_2 + T)](n_3 + T) = [(n_1 n_2) + T](n_3 + T) = (n_1 n_2) n_3 + T =$

$n_1 (n_2 n_3) + T = (n_1 + T)[(n_2 n_3) + T] = (n_1 + T)[(n_2 + T)(n_3 + T)]$

for $n_1, n_2, n_3 \in N$.

Moreover, N/T has the right distributive property, since

$[(n_1 + T) + (n_2 + T)](n_3 + T) = [(n_1 + n_2) + T](n_3 + T) =$

$(n_1 + n_2) n_3 + T = (n_1 n_3 + n_2 n_3) + T = (n_1 n_3 + T) + (n_2 n_3 + T) =$

$(n_1 + T)(n_3 + T) + (n_2 + T)(n_3 + T)$.

This suffices to prove N/T is a near-ring. Hence π may be considered a near-ring homomorphism, from which it follows that $T = \text{Ker } \pi$ is a two-sided ideal of N.

<u>Remark 1.21</u>. If one includes property (d) in Definition 1.1, then one can add to Theorem 1.20 the property $NT \subseteq T$. This can be seen by choosing $b \in NT$, and noting that this implies $b = \sum_{i=1}^{j} n_i t_i$ where $n_i \in N$ and $t_i \in T$. It follows that if $T = \text{Ker } f$, then $f(b) = f(\sum_{i=1}^{j} n_i t_i) = \sum_{i=1}^{j} f(n_i) f(t_i) = \sum_{i=1}^{j} f(n_i) 0 = 0$. This implies $b \in T$ and hence $NT \subseteq T$. However, one should note that the inclusion of this property eliminates the need for 1.20 (b).

Using essentially the same argument as that of Theorem 1.20, we have the following result.

<u>Corollary 1.22</u>. If I is an ideal of the near-ring N, then N/I is a near-ring, where addition is defined by $(a + I) + (b + I) = (a + b) + I$ and multiplication is defined by $(a + I)(b + I) = ab + I$.

Proposition 1.23. If N is a near-ring and if I is a subset of N, then N/I is a near-ring under the definition in Corollary 1.22 if and only if I is an ideal of N.

Proof: $(\rightarrow)$ Suppose N/I is a near-ring. Then $I = \text{Ker } \pi$ where π is the canonical projection from N to N/I defined by $\pi(n) = n + I$ for $n \in N$. Thus by Definition 1.19, I is an ideal of N.

$(\leftarrow)$ See Corollary 1.22.

Definition 1.24. If N is a near-ring and if I is an ideal of N, then N/I is called the factor or quotient near-ring of N by I.

Factor Theorem for Near-Ring Homomorphisms 1.25. Let M and N be near-rings, and let f, an element of $\text{Hom}(M, N)$, be onto. Let I be an ideal of M with $I \subseteq \text{Ker } f$; then there exists a unique onto near-ring homomorphism $\hat{f}$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\pi_I"'] & & N \\ & M/I \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

commutes.

Proof: Let M and N be near-rings, and let f be a homomorphism from M onto N. Suppose I is an ideal of M with $I \subseteq \text{Ker } f$. Define $\hat{f} : M/I \rightarrow N$ by $\hat{f}(a + I) = f(a)$. To show $\hat{f}$ is well-defined, assume $a + I = b + I$. Then $a - b \in I$, and since $I \subseteq \text{Ker } f$, $f(a - b) = 0$. Thus $f(a) = f(b)$ and $\hat{f}(a + I) = \hat{f}(b + I)$.

Next let us show that $\hat{f}$ is a homomorphism from M/I to N. Since $\hat{f}[(a + I) + (b + I)] = \hat{f}[(a + b) + I]$, if follows that $\hat{f}[(a + I) + (b + I)] = f(a + b)$ which equals $f(a) + f(b)$.

But $f(a) + f(b) = \hat{f}(a + I) + \hat{f}(b + I)$. Therefore

$\hat{f}[(a + I) + (b + I)] = \hat{f}(a + I) + \hat{f}(b + I)$. Moreover,

$\hat{f}[(a + I)(b + I)] = \hat{f}(ab + I) = f(ab)$. Since f is a homomorphism, $f(ab) = f(a) f(b)$, which is just $\hat{f}(a + I)\hat{f}(b + I)$. Thus $\hat{f}[(a + I)(b + I)] = \hat{f}(a + I)\hat{f}(b + I)$, and so by Definition 1.11, $\hat{f}$ is a homomorphism from M/I to N.

To show $\hat{f}$ is onto, let $a \in N$. Since f is onto, there exists $b \in M$ such that $f(b) = a$. But $\hat{f}(b + I) = f(b) = a$. Hence $\hat{f}$ is onto. In addition, $(\hat{f} \circ \pi_I)(a) = \hat{f}(a + I) = f(a)$ for every $a \in M$, so $\hat{f} \circ \pi_I = f$.

Finally, if F is any other homomorphism making the diagram commute, then

$F(a + I) = (F \circ \pi_I)(a) = f(a) = \hat{f}(a + I)$. Therefore $F = \hat{f}$.

This completes the proof.

First Isomorphism Theorem 1.26. Let M and N be near-rings, and let f, an element of Hom (M, N), be onto. If $I = \text{Ker } f$, then $M/I \cong N$.

Proof: Suppose M and N are near-rings, and $f \in \text{Hom}(M, N)$ is onto. Also suppose $I = \text{Ker } f$. Then by Proposition 1.23, M/I is a near-ring on which addition is defined by $(a + I) + (b + I) = (a + b) + I$ and multiplication is defined by $(a + I)(b + I) = ab + I$. Define $\hat{f} : M/I \to N$ by $\hat{f}(a + I) = f(a)$ for $a \in M$. Then by the Factor Theorem 1.25, $\hat{f}$ is an onto homomorphism. Moreover, suppose $\hat{f}(a + I) = \hat{f}(b + I)$. Then $f(a) = f(b)$, from which it follows that $0 = f(a) - f(b) = f(a - b)$.

Thus $a - b \in \mathrm{Ker}\, f = I$, so $a + I = b + I$. Therefore $\hat{f}$ is one-to-one, and hence is an isomorphism from M/I to N.

Second Isomorphism Theorem 1.27. Let S and T be sub-near-rings of a near-ring N, with S an ideal of N. Then $S \cap T$ is an ideal of T, $S + T$ is a sub-near-ring of N, and $(S + T)/S \cong T/(S \cap T)$.

Proof: Suppose S and T are sub-near-rings of a near-ring N, with S being an ideal of N.

Let us first show that $S \cap T$ is an ideal of T. Let $a, b \in S \cap T$. Then $a - b \in S$ and $a - b \in T$ since both are sub-near-rings. Thus $a - b \in S \cap T$, which implies $S \cap T$ is a subgroup. To show $S \cap T$ is normal, let $t \in T$, $b \in S \cap T$. Since S is an ideal, $t + b - t \in S$ and $bt \in S$. But $t + b - t \in T$ and $bt \in T$ since T is a sub-near-ring. Hence $t + b - t \in S \cap T$, so $S \cap T$ is an additive normal subgroup of T. Also $bt \in S \cap T$ so $(S \cap T)\, T \subseteq S \cap T$. Moreover, for $t_1, t_2 \in T$ and $b \in S \cap T$, $t_2\,(b + t_1) - t_2\, t_1 \in S$ since S is an ideal, and $t_2\,(b + t_1) - t_2\, t_1 \in T$ since T is a sub-near-ring. Hence $t_2\,(b + t_1) - t_2\, t_1 \in S \cap T$. This suffices to show $S \cap T$ is an ideal of T.

To show that $S + T$ is a sub-near-ring of N, let $a, b \in S + T$. Then $a = s_1 + t_1$, $b = s_2 + t_2$ where $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Hence $a + (-b) = (s_1 + t_1) + (-t_2 - s_2) = s_1 + [(t_1 - t_2) - s_2]$. Since S is an ideal, $s_1 + [(t_1 - t_2) - s_2] = s_1 + [s_3 + (t_1 - t_2)]$, which is contained in $S + T$. Therefore $a - b \in S + T$.

Also $ab = (s_1 + t_1)(s_2 + t_2) = s_1 (s_2 + t_2) + t_1 (s_2 + t_2)$. Since S is an ideal, $s_1 (s_2 + t_2) = s_4$ and $t_1 (s_2 + t_2) = s_5 + t_1 t_2$ for some $s_4, s_5 \in S$. Thus $ab = s_4 + s_5 + t_1 t_2 \in S + T$. Therefore $S + T$ is a sub-near-ring of N.

Now define $f : T \to (S + T)/S$ by $f(t) = S + t$. Since for $t_1, t_2 \in T$, $f(t_1 + t_2) = S + (t_1 + t_2) = (S + t_1) + (S + t_2) = f(t_1) + f(t_2)$ and $f(t_1 t_2) = S + (t_1 t_2) = (S + t_1)(S + t_2) = f(t_1) f(t_2)$, f is a homomorphism. Moreover, f is onto, since for $a \in (S + T)/S$, then $a = s_1 + t_1 + S = s_1 + S + t_1 = S + t_1 = f(t_1)$ for $t_1 \in T$, $s_1 \in S$. Also $\text{Ker } f = \{t \in T \mid S + t = S\} = \{t \in T \mid t \in S\} = S \cap T$.

Therefore by the First Isomorphism Theorem,

$T/\text{Ker } f \cong (S + T)/S$, or $T/(S \cap T) \cong (S + T)/S$.

Third Isomorphism Theorem 1.28. Let $S \subseteq T \subseteq N$ and let S and T be ideals of the near-ring N. Then T/S is an ideal of N/S, and $(N/S) / (T/S) \cong N/T$.

Proof: Let $S \subseteq T \subseteq N$ and let S and T be ideals of the near-ring N. Define $g : N/S \to N/T$ by $g(n + S) = n + T$. Then g is well-defined, since for $a, b \in N$, $a + S = b + S$ implies $a - b \in S$. But $S \subseteq T$ so $a + T = b + T$. Since $g[(a + S) + (b + S)] = g[(a + b) + S] = (a + b) + T = (a + T) + (b + T) = g(a + S) + g(b + S)$, and since $g[(a + S)(b + S)] = g(ab + S) = ab + T = (a + T)(b + T) = g(a + S)\, g(b + S)$, g is a homomorphism. To see that g is onto, let $x \in N/T$. Then $x = n + T$ where $n \in N$. Therefore there exists $y \in N/S$ such that $y = n + S$

and $g(y) = g(n + S) = n + T = x$. Hence g is onto.

Furthermore, $\text{Ker } g = T/S$ since $\text{Ker } g = \{n + S \in N/S \mid n + T = T\}$ $= \{n + S \in N/S \mid n \in T\} = T/S$. Hence T/S is an ideal of N/S, and, by the First Isomorphism Theorem, $(N/S) / (T/S) \cong N/T$.

Correspondence Theorem 1.29. Let N be a near-ring, and let I be an ideal of N. There exists a one-to-one correspondence between the set $\mathcal{S}$ of sub-near-rings of N/I and the set $\mathcal{I}$ of sub-near-rings of N containing I. One defines the correspondences

$H: \mathcal{S} \to \mathcal{I}$ and $\Psi: \mathcal{I} \to \mathcal{S}$ as follows:

Let S be an element of $\mathcal{S}$; then $S \xrightarrow{H} \pi_I^{-1}(S)$, an element of $\mathcal{I}$.

Let L be an element of $\mathcal{I}$; then $L \xrightarrow{\Psi} L/I$, an element of $\mathcal{S}$.

Furthermore, Ψ preserves set inclusions; and for K, L, elements of $\mathcal{I}$, K is an ideal of L if and only if K/I is an ideal of L/I.

Proof: Let N be a near-ring and let I be an ideal of N. Further suppose $S \in \mathcal{S}$ and $a, b \in \pi_I^{-1}(S)$, where $\mathcal{S}$ is the set of sub-near-rings of N/I. Then $\pi_I(ab) = \pi_I(a)\,\pi_I(b) \in S$, which implies $ab \in \pi_I^{-1}(S)$. Also $\pi_I(a - b) = \pi_I(a) - \pi_I(b) \in S$, from which it follows that $a - b \in \pi_I^{-1}(S)$. Thus $\pi_I^{-1}(S)$ is a sub-near-ring of N, and since $I = \text{Ker } \pi_I \subseteq \pi_I^{-1}(S)$, $\pi_I^{-1}(S)$ is contained in $\mathcal{I}$, where $\mathcal{I}$ is the set of sub-near-rings of N containing I.

Let $L \in \mathcal{I}$. Since I is an ideal of N, I is an ideal of L, and hence $L/I \in \mathcal{S}$. Define Ψ from $\mathcal{I}$ to $\mathcal{S}$ by $\Psi(L) = L/I$, and define H from $\mathcal{S}$ to $\mathcal{I}$ by $H(S) = \pi_I^{-1}(S)$. Then $\Psi \circ H(S) = \Psi(\pi_I^{-1}(S)) = \pi_I(\pi_I^{-1}(S)) = S$. Hence $\Psi \circ H = 1_{\mathcal{S}}$.

Also $H \circ \Psi (L) = H(\pi_I (L)) = \pi_I^{-1} (\pi_I (L)) = L$, so $H \circ \Psi = 1_{\mathcal{I}}$. Therefore Ψ is one-to-one and onto, from which it follows that there exists a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{I}$.

To show that Ψ preserves set inclusions, let us suppose that K and L are elements of $\mathcal{I}$ with $K \subseteq L$. Let $x \in K/I$. Then $x = k + I$ where $k \in K$. But since $K \subseteq L$, $k \in L$ which implies $x = k + I \in L/I$. Thus $K/I \subseteq L/I$. Now assume $K/I \subseteq L/I$ and let $k \in K$. Then $k + I \in K/I$, but since $K/I \subseteq L/I$, it follows that $k + I \in L/I$. Hence $k \in L$ which implies $K \subseteq L$. Therefore $K \subseteq L$ if and only if $K/I \subseteq L/I$.

To complete the theorem, assume K is an ideal of L. Then $K = \text{Ker } f$, where f is a homomorphism from L to some near-ring M. Define $\hat{f}$ from L/I to M by $\hat{f}(\ell + I) = f(\ell)$. Then by the Factor Theorem, $\hat{f}$ is a homomorphism.

Moreover, for $k \in K$, $\hat{f}(k + I) = f(k) = 0$, and if $\hat{f}(\ell + I) = 0$, then $f(\ell) = 0$ which implies $\ell \in K$. Hence $\text{Ker } \hat{f} = K/I$, and K/I ia an ideal of L/I. To show the converse, assume K/I is an ideal of L/I. Then $K/I = \text{Ker } g$ where g is a homomorphism from L/I to some near-ring R. Define $\hat{g}$ from L to R by $\hat{g}(\ell) = g(\ell + I)$. Then for all $k \in K$, $\hat{g}(k) = g(k + I) = 0$. If $\hat{g}(\ell) = 0$, then $g(\ell + I) = 0$ which implies $\ell + I \in K/I$, so $\ell \in K$. Thus $K = \text{Ker } \hat{g}$, from which it follows that K is an ideal of L. Hence K is an ideal of L if and only if K/I is an ideal of L/I. This completes the proof of the theorem.

<u>Definition 1.30</u>. Let N be a right near-ring. A <u>left</u> <u>N-module</u>,

${}_N M$, is a (not necessarily abelian) group $(M, +)$ together with a function $\mu : N \times M \to M$, where we denote $\mu(n, m)$ by $n \mu m$, such that:

(a) $(r + s) \mu a = r \mu a + s \mu a$;

(b) $(rs) \mu a = r \mu (s \mu a)$ for all $r, s \in N$ and $a \in M$.

If N is a unitary near-ring, then also

(c) $1 \mu a = a$ for all $a \in M$.

The reader can easily see how to define a right N-module, M_N, where $\mu : M \times N \to N$, for a left near-ring N. Generally, the function μ is suppressed, and $r \mu a$ is written ra. An example of a left N-module is N^+, denoted by ${}_N N$, where the function μ is multiplication on N. Note that it is not necessarily true that $r \mu (a + b) = r \mu a + r \mu b$ for $r \in N$ and $a, b \in M$, since the left distributive property does not necessarily hold for N, and we want ${}_N N$ to be a left N-module. By "module" we shall mean a "left module".

Definition 1.31. Let ${}_N M$ and ${}_N L$ be modules. A function $f : M \to L$ is called an N-homomorphism provided:

$f(a + b) = f(a) + f(b)$ and $f(ra) = rf(a)$ for all $r \in N$ and $a, b \in M$.

Definition 1.32. Submodules of a module ${}_N M$ are defined to be kernels of N-homomorphisms.

Theorem 1.33. A subset A of a module ${}_N M$ is a submodule if and only if

(a) $(A, +)$ is a normal subgroup of $(M, +)$ and

(b) $n(a + m) - n m \in A$ for all $a \in A$, $m \in M$, and $n \in N$.

Proof: $(\rightarrow)$ Suppose a subset A of ${}_N M$ is a submodule. Then A is the kernel of some N-homomorphism g, and hence A is a normal subgroup of M. Also for $a \in A$, $m \in M$, and $n \in N$, $g[n(a + m) - nm] = g[n(a + m)] - g(nm) = n[g(a) + g(m)] - ng(m) = ng(m) - ng(m) = 0$. Hence $n(a + m) - n m \in A$.

$(\leftarrow)$ Suppose A is a subset of ${}_N M$ such that $(A, +)$ is a normal subgroup of $(M, +)$ and such that $n(a + m) - n m \in A$ for all $a \in A$, $m \in M$, and $n \in N$. Let π be the natural group homomorphism from M onto M/A.

Claim: If $\mu : N \times M/A \rightarrow M/A$ is defined by $\mu(n, m + A) = n\,\mu(m + A) = n m + A$, then M/A is a module.

Subproof: To show μ is well-defined, suppose $(r, b + A) = (s, c + A)$ for $r, s \in N$ and $b, c \in M$. It follows that $r = s$ and $b + A = c + A$. Hence $b = c + a_1$ for some $a_1 \in A$. So $rb = s(c + a_1)$, and since $(A, +)$ is a normal subgroup of $(M, +)$, $s(c + a_1) = s(a_2 + c)$ for some $a_2 \in A$. But $s(a_2 + c) - s c \in A$ so $s(a_2 + c) - s c = a_3$ for some $a_3 \in A$. This implies $s(a_2 + c) = a_3 + s c$ or $s(a_2 + c) = sc + a_4$ for some $a_4 \in A$. Hence $rb = sc + a_4$, so $rb + A = sc + A$, from whence it follows that μ is well-defined.

Moreover, for $r, s \in N$, $m \in M$, $(r + s)\,\mu\,(m + A) = (r + s)\,m + A = (rm + sm) + A$. But $(rm + sm) + A = (rm + A) + (sm + A) = r\mu(m + A) + s\mu(m + A)$. Hence $(r + s)\,\mu\,(m + A) = r\mu\,(m + A) + s\mu\,(m + A)$.

Also $(rs)\,\mu\,(m + A) = (rs)\,m + A = r(sm) + A = r\,\mu\,[(sm) + A] =$

$r\mu\,[s\mu\,(m + A)]$. Therefore M/A is a module, and π is an N-homomorphism, which makes $A = \text{Ker}\,\pi$ a submodule of ${}_N M$.

<u>Corollary 1.34</u>. Let ${}_N M$ be a module, and let A be a submodule of M. Then M/A is an N-module, where $n\,(m + A) = nm + A$ for all $n \in N$, $m \in M$.

<u>Definition 1.35</u>. Let N be a near-ring; submodules of ${}_N N$ are called <u>left-ideals</u>. If A is a submodule of the module ${}_N M$, we call ${}_N(M/A)$ the <u>factor</u> or <u>quotient</u> <u>module</u>.

Note that Theorem 1.33 gives a characterization of the left ideals of a near-ring. Hence we see that the additive normal subgroups of a near-ring N satisfying property (b) of Theorem 1.20 are left ideals of N; those satisfying property (a) are right-ideals.

<u>Definition 1.36</u>. A subgroup $B = (B, +)$ of a module ${}_N M$ is called a <u>left</u> <u>N-subgroup</u> if $NB \subseteq B$.

Note that if one includes property (d) in Definition 1.1, then submodules are N-subgroups.

CHAPTER II

NEAR-RING EMBEDDINGS

In this chapter, we shall consider certain embeddings of groups into near-rings, both with and without identities. Many of these results come from the paper by Malone and Heatherly [7], and the paper by Beidleman [1]. Throughout the chapter, G and H will denote nontrivial additive groups, with their identities denoted by 0. Recall from Chapter I that $T(G)$ is the set of all mappings on G.

Definition 2.1. Let $T_0(G) = \{f \in T(G) \mid f(0) = 0\}$. Then it can be shown that $T_0(G)$ is a sub-near-ring of $T(G)$.

Definition 2.2. Let $g \in G$, and let ϕ_g be defined by $\phi_g(a) = g$ for all $a \in G$. Then $T'(G) = \{\phi_g \mid g \in G\}$ can be shown to be a sub-near-ring of $T(G)$, since $\phi_g - \phi_h = \phi_{g-h}$ and $\phi_g \phi_h = \phi_g$.

Definition 2.3. For $g \in G$, define

$$\Psi_g(a) = \begin{cases} g & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases} \quad \text{for all } a \in G.$$

Then $\Psi_g \in T_0(G)$. Let $T_0'(G) = \{\Psi_g \in T_0(G) \mid g \in G\}$. Since $\Psi_g - \Psi_h = \Psi_{g-h}$ and $\Psi_g \Psi_h = \begin{cases} \Psi_g & \text{if } h \neq 0 \\ 0 & \text{if } h = 0, \end{cases}$

it can be shown that $T_0'(G)$ is a sub-near-ring of $T_0(G)$.

Proposition 2.4. Define f from G to $T'(G)$ by $f(g) = \phi_g$ for all $g \in G$. Then f is an isomorphism of the additive group G onto the additive group of $T'(G)$.

Proof: Suspose f is a mapping from G to $T'(G)$ defined by $f(g) = \phi_g$ for all $g \in G$. For $g, h \in G$, $f(g + h) = \phi_{g+h} = \phi_g + \phi_h = f(g) + f(h)$. Hence f is a group homomorphism. Also $f(g) = f(h)$ implies $\phi_g = \phi_h$, from which it follows that $g = h$. Moreover, if $\phi_g \in T'(G)$, then $g \in G$ so $f(g) = \phi_g$. Therefore f is a group isomorphism from G to $T'(G)$.

Proposition 2.5. Define f from G to $T_0'(G)$ by $f(g) = \Psi_g$ for all $g \in G$. Then f is an isomorphism of the additive group G onto the additive group of $T_0'(G)$.

Proof: Suppose f is a mapping from G to $T_0'(G)$ defined by $f(g) = \Psi_g$ for all $g \in G$. Then f is a group homomorphism since $f(g + h) = \Psi_{g+h} = \Psi_g + \Psi_h = f(g) + f(h)$ for all $g, h \in G$. Suppose $f(g) = f(h)$. Then $\Psi_g(a) = \Psi_h(a)$ for all $a \in G$. Since G is nontrivial, we may let $a \neq 0$. Then $\Psi_g(a) = \Psi_h(a)$ implies $g = h$. Hence f is one-to-one. To complete the proof, let $\Psi_g \in T_0'(G)$. Then $g \in G$, so $f(g) = \Psi_g$. Therefore f is a group isomorphism from G onto the additive group of $T_0'(G)$.

Theorem 2.6. Let N be a near-ring. If G is any additive group containing N^+ as a proper subgroup, then N can be embedded in $T(G)$.

Proof: Let N be a near-ring, and let G be any additive group containing N^+ as a proper subgroup. Let $a \in N$, and define τ_a from G into G by

$$\tau_a(x) = \begin{cases} ax & \text{if } x \in N \\ a & \text{if } x \in G \backslash N. \end{cases}$$

Let $A = \{\tau_a \in T(G) \mid a \in N\}$. Let us show that A is a sub-near-ring of $T(G)$. If $x \in N$, then

$\tau_a \tau_b (x) = \tau_a (\tau_b (x)) = \tau_a (bx) = abx = \tau_{ab} (x)$, and

$(\tau_a + \tau_b) (x) = \tau_a (x) + \tau_b (x) = ax + bx = (a + b)x = \pi_{a+b} (x)$.

If $x \in G \backslash N$, then

$(\tau_a \tau_b) (x) = \tau_a (\tau_b (x)) = \tau_a (b) = ab = \tau_{ab} (x)$, and

$(\tau_a + \tau_b) (x) = \tau_a (x) + \tau_b (x) = a + b = \tau_{a+b} (x)$. Hence $\tau_a \tau_b = \tau_{ab}$ and $\tau_a + \tau_b = \tau_{a+b}$, from which it follows that A is closed under addition and multiplication. If $x \in N$, then $- \tau_a (x) = - (ax) = (-a) (x) = \tau_{-a} (x)$, and if $x \in G\backslash N$, then $-\tau_a (x) = -a = \tau_{-a} (x)$. Hence $-\tau_a = \tau_{-a}$. Therefore $\tau_a - \tau_b = \tau_a + \tau_{-b} = \tau_{a-b} \in A$, and $\tau_a \tau_b = \tau_{ab} \in A$. Thus A is a sub-near-ring of $T(G)$.

Define f from N to $T(G)$ by $f(a) = \tau_a$. Then f is a near-ring homomorphism since

$f(a + b) = \tau_{a+b} = \tau_a + \tau_b = f(a) + f(b)$, and

$f(ab) = \tau_{ab} = \tau_a \tau_b = f(a) f(b)$.

Now suppose $\tau_a = \tau_b$. Since N is a proper subgroup of G, there exists an $x \in G\backslash N$. Hence $\tau_a (x) = \tau_b (x)$, from which it follows that $a = b$. Thus f is a monomorphism and is the desired embedding mapping of N into $T(G)$.

Corollary 2.7. Every near-ring can be embedded in a near-ring with identity.

Proof: Let 1_G denote the function from G to G defined by $1_G (g) = g$ for all $g \in G$. Then 1_G is the identity for $T(G)$, so the corollary follows immediately from Theorem 2.6.

<u>Definition 2.8</u>. Let G and H be groups, and let f be a mapping from G to H. The <u>kernel of f</u>, denoted Ker f, is defined as follows:

Let 0 be the identity of H; then $\text{Ker } f = \{a \in G \mid f(a) = 0\}$. The function f is said to be <u>kernel-free</u> if $\text{Ker } f = \{0\}$. A near-ring homomorphism from $T_0(G)$ into $T_0(H)$ is called <u>kernel-preserving</u> if it maps kernel-free elements of $T_0(G)$ onto the kernel-free elements of $T_0(H)$.

<u>Proposition 2.9</u>. The non-zero elements of $T_0'(G)$ are kernel-free mappings of G into G.

<u>Proof</u>: Let $\Psi_g \in T_0'(G)$ where $g \neq 0$. If $h \in \text{Ker } \Psi_g$, then $\Psi_g(h) = 0$ which implies $h = 0$. Hence $\text{Ker } \Psi_g = \{0\}$.

<u>Proposition 2.10</u>. Let $\gamma \in T_0(G)$. Then γ is kernel-free if and only if $\Psi_g \circ \gamma = \Psi_g$ for every non-zero element g of G.

<u>Proof</u>: Let $\gamma \in T_0(G)$, and suppose γ is kernel-free. Let $a \in G$. If $a \neq 0$, then $\gamma(a) \neq 0$ and $(\Psi_g \circ \gamma)(a) = \Psi_g(\gamma(a)) = \Psi_g(a)$. If $a = 0$, then $(\Psi_g \circ \gamma)(a) = \Psi_g(\gamma(a)) = \Psi_g(0) = 0$. Hence $\Psi_g \circ \gamma = \Psi_g$.

Conversely, suppose $\Psi_{g \circ \gamma} = \Psi_g$ for every $g \in G$ where $g \neq 0$. Let $a \in \text{Ker } \gamma$. Then $(\Psi \circ \gamma)(a) = \Psi_g(\gamma(a)) = \Psi_g(0) = 0$. But $(\Psi_g \circ \gamma)(a) = \Psi_g(a)$ by hypothesis. Hence $\Psi_g(a) = 0$ which implies $a = 0$. Therefore $\text{Ker } \gamma = \{0\}$, from which it follows that γ is kernel-free.

<u>Lemma 2.11</u>. Let G and H be groups, and let f be an isomorphism of G onto H. Then f induces a near-ring isomorphism

of T(G) onto T(H).

Proof: Let $\gamma \in T(G)$. Then $f \circ \gamma \circ f^{-1}$ is contained in T(H). Define μ from T(G) into T(H) by $\mu(\gamma) = f \circ \gamma \circ f^{-1}$. Then μ is a near-ring homomorphism since for $\alpha, \gamma \in T(G)$, $\mu(\alpha + \gamma) =$

$f \circ (\alpha + \gamma) \circ f^{-1} = f \circ [\alpha \circ f^{-1} + \gamma \circ f^{-1}] = f \circ \alpha \circ f^{-1} + f \circ \gamma \circ f^{-1}$

$= \mu(\alpha) + \mu(\gamma)$, and

$\mu(\alpha \circ \gamma) = f \circ (\alpha \circ \gamma) \circ f^{-1} = (f \circ \alpha \circ f^{-1}) \circ (f \circ \gamma \circ f^{-1}) =$

$\mu(\alpha) \circ \mu(\gamma)$.

To show that μ is one-to-one, let us assume $\mu(\alpha) = \mu(\gamma)$. Then $f \circ \alpha \circ f^{-1} = f \circ \gamma \circ f^{-1}$, which implies $f^{-1} \circ f \circ \alpha \circ f^{-1} \circ f =$ $f^{-1} \circ f \circ \gamma \circ f^{-1} \circ f$ or $\alpha = \gamma$. Also μ is onto, since for $\beta \in T(H)$, $f^{-1} \circ \beta \circ f \in T(G)$, so $\mu(f^{-1} \circ \beta \circ f) =$

$f \circ (f^{-1} \circ \beta \circ f) \circ f^{-1} = \beta$. Hence f induces the near-ring isomorphism μ from T(G) to T(H).

Proposition 2.12. Let f be a group isomorphism of G onto H. Then

(i) f induces a near-ring isomorphism of T(G) onto T(H);

(ii) f induces a near-ring isomorphism of $T_0(G)$ onto $T_0(H)$;

(iii) f induces a near-ring isomorphism of T'(G) onto T'(H);

(iv) f induces a near-ring isomorphism of $T_0'(G)$ onto $T_0'(H)$.

Proof: (i) follows from Lemma 2.11. To show (ii), define μ from $T_0(G)$ into $T_0(H)$ by $\mu(\beta) = f \circ \beta \circ f^{-1}$ for all $\beta \in T_0(G)$. Then $f \circ \beta \circ f^{-1} \in T(H)$. Moreover, $(f \circ \beta \circ f^{-1})(0)$

$= f(\beta(f^{-1}(0))) = f(\beta(0)) = f(0) = 0$, so $f \circ \beta \circ f^{-1} \in T_0(H)$.

By the proof of Lemma 2.11, μ is a one-to-one near-ring homomorphism.

To show μ is onto, let $\alpha \in T_0(H)$. Then $f^{-1} \circ \alpha \circ f \in T_0(G)$, so $\mu\,(f^{-1} \circ \alpha \circ f) = f\,(f^{-1} \circ \alpha \circ f)\,f^{-1} = \alpha$. Hence μ is a near-ring isomorphism from $T_0(G)$ to $T_0(H)$.

To prove (iii), define μ from $T'(G)$ into $T'(H)$ by $\mu\,(\phi_g) = f \circ \phi_g \circ f^{-1}$ for $\phi_g \in T'(G)$. Then $f \circ \phi_g \circ f^{-1} \in T(H)$. Moreover, for $h \in H$, $(f \circ \phi_g \circ f^{-1})\,(h) = f\,(\phi_g\,(f^{-1}\,(h))) = f(g)$ $= \phi_{f(g)}\,(h)$, so $\mu\,(\phi_g) = \phi_{f(g)} \in T'(H)$. By the proof of Lemma 2.11, μ is a one-to-one near-ring homomorphism. To show μ is onto, let $\phi_h \in T'(H)$. Then $h \in H$, and since f is an isomorphism from G to H, there exists a $g \in G$ such that $f(g) = h$. Hence $\mu\,(\phi_g) = \phi_{f(g)} = \phi_h$. Therefore μ is a near-ring isomorphism from $T'(G)$ onto $T'(H)$.

To prove (iv), define μ from $T_0'(G)$ to $T_0'(H)$ by $\mu\,(\Psi_g) = f \circ \Psi_g \circ f^{-1}$. Then by (ii), $f \circ \Psi_g \circ f^{-1} \in T_0(H)$, and for $h \in H$, $h \neq 0$, $(f \circ \Psi_g \circ f^{-1})\,(h) = f\,(\Psi_g\,(f^{-1}\,(h))) = f(g)$. Moreover, $(f \circ \Psi_g \circ f^{-1})\,(0) = f\,(\Psi_g\,(f^{-1}\,(0))) = f\,(\Psi_g\,(0)) = 0$. Hence $\mu\,(\Psi_g) = f \circ \Psi_g \circ f^{-1} = \Psi_{f(g)} \in T_0'(H)$. By the proof of Lemma 2.11, μ is a one-to-one near-ring homomorphism. To show μ is onto, let $\Psi_h \in T_0'(H)$. Then $h \in H$, and since f is an isomorphism from G to H, there exists $g \in G$ such that $f(g) = h$. Therefore $\mu\,(\Psi_g) = \Psi_{f(g)} = \Psi_h$. Thus μ is a near-ring isomorphism from $T_0'(G)$ to $T_0'(H)$. This suffices to prove the theorem.

<u>Proposition 2.13</u>. If $T(H)$ is a near-ring homomorphic image of $T(G)$, then H is a homomorphic image of G.

<u>Proof</u>: Suppose $T(H)$ is a near-ring homomorphic image of $T(G)$. Then there exists a near-ring homomorphism f from $T(G)$ onto $T(H)$.

Let $\phi_g \in T'(G)$ and $\phi_h \in T'(H)$. Let us first show that f maps $T'(G)$ into $T'(H)$. Since $T'(H) \subseteq T(H)$, there exists $\gamma \in T(G)$ such that $f(\gamma) = \phi_h$. Hence

$$f(\phi_g) = f(\phi_g \circ \gamma) = (f(\phi_g)) \circ (f(\gamma)) = (f(\phi_g)) \circ (\phi_h).$$ But

$f(\phi_g) = \alpha$ where $\alpha \in T(H)$. Hence for $a \in H$,

$$(f(\phi_g)) \circ (\phi_h)(a) = \alpha(\phi_h(a)) = \alpha(h) = \phi_{\alpha(h)}(a).$$ Therefore

$f(\phi_g) = \phi_{\alpha(h)} \in T'(H)$. To show f is onto, we need only note that $f(\phi_{\gamma(g)}) = f(\gamma \circ \phi_g) = f(\gamma) \circ f(\phi_g) = \phi_h \circ (f(\phi_g)) = \phi_h$. Hence f is a near-ring homomorphism from $T'(G)$ onto $T'(H)$, so by Proposition 2.4, H is the homomorphic image of G.

Theorem 2.14. Let G and H be groups. The following are equivalent:

(i) there exists a group isomorphism from G onto H;

(ii) there exists a near-ring isomorphism from $T(G)$ onto $T(H)$;

(iii) there exists a near-ring isomorphism from $T'(G)$ onto $T'(H)$.

Proof: (i)→(ii) If G is group isomorphic to H, then by Lemma 2.11, $T(G)$ is isomorphic to $T(H)$.

(ii)→(iii) By the proof of Proposition 2.13, if f is a near-ring isomorphism of $T(G)$ onto $T(H)$, then $f[T'(G)] = T'(H)$. Hence f is a near-ring isomorphism of $T'(G)$ onto $T'(H)$.

(iii)→(i) Suppose $T'(G)$ is near-ring isomorphic to $T'(H)$. Then $T'(G)$ is group isomorphic to $T'(H)$, so by Proposition 2.4, G is group isomorphic to H.

Theorem 2.15. Let G and H be groups. The following are equivalent:

(i) there exists a group isomorphism from G onto H;

(ii) there exists a kernel-preserving near-ring isomorphism from $T_0(G)$ onto $T_0(H)$;

(iii) there exists a kernel-preserving near-ring isomorphism from $T_0'(G)$ onto $T_0'(H)$.

<u>Proof</u>: (i)→(ii) Suppose there exists a group isomorphism f from G onto H. By Theorem 2.12, f induces a near-ring isomorphism μ from $T_0(G)$ onto $T_0(H)$ defined by $\mu(\beta) = f \circ \beta \circ f^{-1}$ for $\beta \in T_0(G)$. To show μ is kernel-preserving, let us suppose β is a kernel-free element of $T_0(G)$. Let $a \in \text{Ker } f \circ \beta \circ f^{-1}$. Since f is a group isomorphism, $\beta(f^{-1}(a)) = 0$. But β is kernel-free, so $f^{-1}(a) = 0$, from which it follows that $a = 0$. Further, μ maps the kernel-free elements of $T_0(G)$ onto the kernel-free elements of $T_0(H)$, since if $\alpha \in T_0(H)$ is kernel-free, then $f^{-1} \circ \alpha \circ f \in T_0(G)$ is kernel-free and $\mu(f^{-1} \circ \alpha \circ f) = \alpha$. Hence μ is kernel-preserving.

(ii)→(iii) Suppose there exists a kernel-preserving near-ring isomorphism f from $T_0(G)$ onto $T_0(H)$. Since $T_0'(G) \subseteq T_0(G)$, then $f[T_0'(G)] \subseteq T_0(H)$. Let Ψ_g be a non-zero element of $T_0'(G)$, and let Ψ_h be a non-zero element of $T_0'(H)$. Then by Proposition 2.9, Ψ_h and Ψ_g are kernel-free. Since $T_0'(H) \subseteq T_0(H)$, and since f is a kernel-preserving isomorphism, there exists a kernel-free element $\gamma \in T_0(G)$ such that $f(\gamma) = \Psi_h$.

To show that $f[T_0'(G)] \subseteq T_0'(H)$, we note that $f(\Psi_g) = f(\Psi_g \circ \gamma)$ by Proposition 2.10. Hence, since f is kernel-preserving

$f(\Psi_g) = f(\Psi_g \circ \gamma) = f(\Psi_g) \circ f(\gamma) = f(\Psi_g) \circ \Psi_h = \Psi_{[f(\Psi g)](h)} \in T_0'(H)$.
Finally to show that $f[T_0'(G)] = T_0'(H)$, we remark that
$f(\Psi_{\gamma(g)}) = f(\gamma \circ \Psi_g) = f(\gamma) \circ f(\Psi_g) = \Psi_h \circ f(\Psi_g) = \Psi_h$ since f is kernel-preserving.

(iii) $\rightarrow$ (i) Suppose $T_0'(G)$ is isomorphic as a near-ring to $T_0'(H)$. Then by Proposition 2.5, G is group isomorphic to H.

Theorem 2.16. A nontrivial group G can be embedded in a group H if and only if $T_0(G)$ can be embedded in $T_0(H)$ by a near-ring monomorphism which is kernel-preserving.

Proof: $(\rightarrow)$ Suppose the group G can be embedded in the group H. If the image of G is equal to H, then by Theorem 2.15, there exists a near-ring isomorphism from $T_0(G)$ onto $T_0(H)$ which is kernel-preserving.

Therefore assume that the image of G is a proper subgroup of H, and identify G with this image. Let g be an arbitrary but fixed non-zero element of G. For each $\alpha \in T_0(G)$, define $\alpha' \in T_0(H)$ by

$$\alpha'(x) = \begin{cases} \alpha(x) & \text{if } x \in G \\ \alpha(g) & \text{if } x \in H \backslash G. \end{cases}$$

If α is kernel-free, then $\alpha(g) \neq 0$, so α' is also kernel-free.

Define μ from $T_0(G)$ to $T_0(H)$ by $\mu(\alpha) = \alpha'$. Then μ is obviously well-defined and one-to-one. To show μ is a near-ring homomorphism, let us consider $\mu(\alpha + \beta)$ and $\mu(\alpha \circ \beta)$ where $\alpha, \beta \in T_0(G)$. If $x \in G$, then

$[\mu(\alpha + \beta)](x) = (\alpha + \beta)'(x) = (\alpha + \beta)(x) = \alpha(x) + \beta(x) = \alpha'(x) + \beta'(x)$
$= (\alpha' + \beta')(x)$, and

$[\mu(\alpha \circ \beta)]\ (x) = (\alpha \circ \beta)'\ (x) = (\alpha \circ \beta)\ (x) = \alpha(\beta(x)) = \alpha'\ (\beta'(x))$
$= (\alpha' \circ \beta')\ (x)$. If $x \in H\backslash G$, then
$[\mu(\alpha + \beta)]\ (x) = (\alpha + \beta)'\ (x) = (\alpha + \beta)\ (g) = \alpha(g) + \beta(g)$
$= \alpha'\ (x) + \beta'\ (x) = (\alpha' + \beta')\ (x)$, and
$[\mu(\alpha \circ \beta)]\ (x) = (\alpha \circ \beta)'\ (x) = (\alpha \circ \beta)\ (g) = \alpha(\beta(g)) = \alpha'\ (\beta'\ (x))$
$= (\alpha' \circ \beta')\ (x)$. Hence $\mu(\alpha + \beta) = \mu(\alpha) + \mu(\beta)$ and $\mu(\alpha \circ \beta) = \mu(\alpha) \circ \mu(\beta)$. Thus $T_0(G)$ can be embedded in $T_0(H)$ by the kernel-preserving near-ring monomorphism μ.

($\leftarrow$) Let $T_0(G)$ be embedded in $T_0(H)$ by a kernel-preserving near-ring monomorphism μ. Let a be an arbitrary but fixed non-zero element of H. Define v from $T_0'(G)$ into $T_0'(H)$ by $v\ (\Psi_g) = \Psi_{\alpha(a)}$ where $\alpha = \mu(\Psi_g)$. Let us show by contradiction that v is one-to-one. If $g, k \in G$ with $g \neq k$, then $\Psi_g \neq \Psi_k$. Assume $v(\Psi_g) = v(\Psi_k)$, so $[\mu\ (\Psi_g)]\ (a) = [\mu\ (\Psi_k)]\ (a)$. Then $[\mu\ (\Psi_g - \Psi_k)]\ (a) = 0$ so $\mu\ (\Psi_{g-k})\ (a) = 0$. Since $g - k \neq 0$, then Ψ_{g-k} is kernel-free. Also μ is kernel-preserving, so it follows that $\mu\ (\Psi_{g-k})$ is kernel-free and $a = 0$. But a is non-zero by supposition. From this contradiction we can conclude that v is injective.

To show v is well-defined, let us assume $\Psi_g = \Psi_k$. Then $\mu\ (\Psi_g) = \mu\ (\Psi_k)$ and $[\mu\ (\Psi_g)]\ (a) = [\mu\ (\Psi_k)]\ (a)$. Hence $\Psi_{\alpha_1\ (a)} = \Psi_{\alpha_2(a)}$ where $\alpha_1\ (a) = \mu\ (\Psi_g)\ (a)$ and $\alpha_2\ (a) = \mu\ (\Psi_k)\ (a)$. Therefore $v\ (\Psi_g) = v\ (\Psi_k)$, which implies v is well-defined.

Finally we need to show that v is a group homomorphism. By definition, $v\ (\Psi_g + \Psi_k) = \Psi_{\alpha(a)}$ where $\alpha\ (a) = [\mu\ (\Psi_g + \Psi_k)]\ (a)$.

But $[\mu\ (\Psi_g + \Psi_k)]\ (a) = [\mu\ (\Psi_g) + \mu\ (\Psi_k)]\ (a) = [\mu\ (\Psi_g)]\ (a)$ $+ [\mu\ (\phi_k)]\ (a)$. This implies $\alpha\ (a) = \alpha_1\ (a) + \alpha_2\ (a)$ where $\alpha_1\ (a) = \mu\ (\Psi_g)\ (a)$ and $\alpha_2\ (a) = \mu\ (\Psi_k)\ (a)$. Therefore $v\ (\Psi_g + \Psi_k) = \Psi_{\alpha_1(a)} + \Psi_{\alpha_2(a)} = v\ (\Psi_g) + v\ (\Psi_k)$. Hence the additive group of $T_0'(G)$ is embedded in the additive group of $T_0'(H)$, so by Proposition 2.5, G can be embedded in H.

<u>Theorem 2.17</u>. A nontrivial group G can be embedded in a group H if and only if the near-ring $T(G)$ can be embedded in the near-ring $T(H)$.

<u>Proof</u>: $(\rightarrow)$ Suppose the group G can be embedded in the group H. If the image of G is equal to H, then by Theorem 2.14, there exists a near-ring isomorphism from $T(G)$ to $T(H)$. Assume then that the image of G is a proper subgroup of H, and identify G with this image. Let g be an arbitrary but fixed non-zero element of G. For each $\alpha \in T(G)$, define $\alpha' \in T(H)$ by

$$\alpha'\ (x) = \begin{cases} \alpha\ (x) & \text{if } x \in G \\ \alpha\ (g) & \text{if } x \in H \backslash G. \end{cases}$$

Consider μ from $T(G)$ to $T(H)$ defined by $\mu\ (\alpha) = \alpha'$. By the proof of Theorem 2.16, μ is a near-ring monomorphism. Hence $T(G)$ is embedded in $T(H)$.

$(\leftarrow)$ Conversely, let $T(G)$ be embedded in $T(H)$ by a near-ring monomorphism μ. Since $T'(G) \subseteq T(G)$, then $\mu\ [T'(G)] \subseteq T(H)$. Hence for $\phi_g \in T'(G)$, $\mu\ (\phi_g) = \alpha$ where $\alpha \in T(H)$. But $\mu\ (\phi_g) = \mu\ (\phi_g \circ \phi_0) = \mu\ (\phi_g) \circ \mu\ (\phi_0) = \alpha \circ \phi_0$. Moreover, $(\alpha \circ \phi_0)\ (a) = \alpha\ (\phi_0\ (a)) = \alpha\ (0) = \phi_{\alpha(0)}\ (a)$ for all $a \in H$, so $\mu(\phi_g) = \phi_{\alpha(0)} \in T'(H)$. Therefore $T'(G)$ is embedded in $T'(H)$, so by Proposition 2.4, G is embedded in H.

CHAPTER III

IDEALS IN TRANSFORMATION NEAR-RINGS

In this chapter, we shall concentrate on a specific near-ring, the transformation near-ring, $T_0(G)$, of all mappings of a group G into itself which map 0 onto 0. Specifically, left ideals of $T_0(G)$ will be considered. These results come from the paper by Heatherly [6]. Throughout the chapter, G will denote an additive group with identity 0.

<u>Definition 3.1</u>. If S is a non-empty subset of a group G, then $A(S) = \{\alpha \in T_0(G) \mid \alpha [S] = 0\}$.

<u>Remark 3.2</u>. We note that $A(S)$ is a left ideal of $T_0(G)$. Choose $\alpha, \beta \in A(S)$ and $\gamma, \mu \in T_0(G)$. Then $(\alpha - \beta) [S] = \alpha [S] - \beta [S] = 0$, so $A(S)$ is a subgroup of $T_0(G)$. Also if $\mu \in \gamma + A(S) - \gamma$, then $\mu = \gamma + \alpha - \gamma$ for some $\alpha \in A(S)$. Hence $\mu [S] = (\gamma + \alpha - \gamma) [S] = \gamma [S] + \alpha [S] - \gamma [S] = \gamma [S] - \gamma [S] = 0$. Therefore $\gamma + A(S) - \gamma \subseteq A(S)$ which implies that $A(S)$ is normal in $T_0(G)$.

Finally, for $\mu, \gamma \in T_0(G)$ and $\alpha \in A(S)$, $[\mu (\alpha + \gamma) - \mu \gamma] [S] = [\mu (\alpha + \gamma)] [S] - (\mu \gamma) [S] = \mu (\alpha [S] + \gamma [S]) - \mu (\gamma [S]) = \mu (\gamma [S]) - \mu (\gamma [S]) = 0$. Hence $\mu (\alpha + \gamma) - \mu \gamma \in A(S)$, so $A(S)$ is a left ideal by Theorem 1.33 and Definition 1.35.

<u>Definition 3.3</u>. Let G be a group, let $0 \neq x \in G$, and let $P_x = A(G - \{x\})$. If $y \in G$, let

$$\alpha\ (x, y)\ (t) = \begin{cases} 0 & \text{if } t \neq x \\ y & \text{if } t = x \text{ for every } t \in G. \end{cases}$$

Then P_x may also be described as $\{\alpha\ (x,y) \mid y \in G\}$.

Proposition 3.4. If $y \neq s \neq 0$, then

(i) $\alpha\ (s, t) \circ \alpha\ (x, y) = \alpha\ (x, 0) = 0$;

(ii) $\alpha\ (s, t) \circ \alpha\ (x, s) = \alpha\ (x, t)$.

Proof: Suppose $y \neq s \neq 0$. To show (i), let $v \in G$. Then if $v \neq x$,

$$[\alpha\ (s, t) \circ \alpha\ (x, y)]\ (v) = \alpha\ (s, t)\ [\alpha\ (x, y)\ (v)] = \alpha\ (s, t)\ (0) = 0.$$

If $v = x$, $[\alpha\ (s, t)\ \alpha\ (x, y)]\ (v) = \alpha\ (s, t)\ [\alpha\ (x, y)\ (v)] =$ $\alpha\ (s, t)\ (y) = 0$. But $\alpha\ (x, 0)\ (v) = 0$ if $x = v$ or if $x \neq v$. Hence $\alpha\ (s, t)\ \alpha\ (x, y) = \alpha\ (x, 0) = 0$.

To show (ii), let $v \in G$. Then if $v \neq x$, $[\alpha\ (s, t)\ \alpha\ (x, s)]\ (v) = \alpha\ (s, t)\ (0) = 0$, and if $v = x$, then $[\alpha\ (s, t)\ \alpha\ (x, s)]\ (v) = \alpha\ (s, t)\ (s) = t$. But if $v \neq x$, $\alpha\ (x, t)\ (v) = 0$, and if $v = x$, then $\alpha\ (x, t)\ (v) = t$. Hence $\alpha\ (s, t)\ \alpha\ (x, s) = \alpha\ (x, t)$. This completes the proof.

Proposition 3.5. Let $E = \{\alpha\ (x, x) \mid x \in G \sim \{0\}\}$. Then E is a collection of pairwise orthogonal idempotents.

Proof: Let $\alpha\ (y, y) \in E$. Then by Proposition 3.4 (ii), $\alpha\ (y, y)^2 = \alpha\ (y, y)\ \alpha\ (y, y) = \alpha\ (y, y)$. Hence $\alpha\ (y, y)$ is idempotent. Moreover if $\alpha\ (z, z), \alpha\ (y, y) \in E$ with $z \neq y$, then by Proposition 3.4 (i), $\alpha\ (z, z)\ \alpha\ (y, y) = \alpha\ (y, 0) = 0$. Therefore elements of E are pairwise orthogonal idempotents.

Proposition 3.6. If $\alpha\ (x, y),\ \alpha\ (x, v) \in P_x$ and $\beta \in T_0(G)$, then:

(i) $\alpha\ (x, y) + \alpha\ (x, v) = \alpha\ (x, y + v)$;

(ii) $-\ \alpha\ (x, y) = \alpha\ (x, -y)$;

(iii) $\alpha\ (x, y) - \alpha\ (x, v) = \alpha\ (x, y - v)$;

(iv) $\beta \circ \alpha\ (x, y) = \alpha\ (x, \beta\ (y))$.

Proof: To show (i), let $t \in G$. Then if $t = x$, $[\alpha\ (x, y) + \alpha\ (x, v)]\ (t) = \alpha\ (x, y)\ (t) + \alpha\ (x, v)\ (t) = y + v$, and if $t \neq x$, $[\alpha\ (x, y) + \alpha\ (x, v)]\ (t) = \alpha\ (x, y)\ (t) + \alpha\ (x, v)\ (t) = 0$. But if $t = x$, $\alpha\ (x, y + v)\ (t) = y + v$, and if $t \neq x$, then $\alpha\ (x, y + v)\ (t) = 0$. Hence $\alpha\ (x, y) + \alpha\ (x, v) = \alpha\ (x, y + v)$.

For the proof of (ii), let $t \in G$. If $t = x$, then $-\alpha\ (x, y)\ (t) = -y$, and if $t \neq x$, $-\alpha\ (x, y)\ (t) = 0$. But if $t = x$, $\alpha\ (x, -y)\ (t) = -y$, and if $t \neq x$, $\alpha\ (x, -y)\ (t) = 0$. Thus $-\alpha\ (x, y) = \alpha\ (x, -y)$.

To show (iii), we need only note that by parts (i) and (ii), $\alpha\ (x, y) - \alpha\ (x, v) = \alpha\ (x, y) + \alpha\ (x, -v) = \alpha\ (x, y - v)$.

Finally, to show (iv), let $t \in G$. If $x = t$, then $[\beta \circ \alpha\ (x, y)]\ (t) = \beta\ [\alpha\ (x, y)\ (t)] = \beta\ (y)$, and if $x \neq t$, then $[\beta \circ \alpha\ (x, y)]\ (t) = \beta\ [\alpha\ (x, y)\ (t)] = \beta\ (0) = 0$. But if $x = t$, $\alpha\ (x, \beta\ (y))\ (t) = \beta\ (y)$, and if $x \neq t$, then $\alpha\ (x, \beta\ (y))\ (t) = 0$. Hence $\beta \circ \alpha\ (x, y) = \alpha\ (x, \beta\ (y))$. This completes the proof.

Theorem 3.7. For each $x \neq 0$, P_x is a minimal left ideal of $T_0(G)$ and a minimal $T_0(G)$-subgroup. Also P_x is generated by the idempotent $\alpha\ (x, x)$, which acts as a right identity for P_x.

Proof: For each $x \neq 0$, P_x is a left ideal of $T_0(G)$ by Remark 3.2. Suppose H is a left ideal of $T_0(G)$ that is contained in P_x. Choose $0 \neq \alpha\ (x, y) \in H$ and $\alpha\ (x, z) \in P_x$. Then $\alpha\ (y, z)\ [\alpha\ (x, y) + \alpha\ (x, 0)] - \alpha\ (y, z)\ \alpha\ (x, 0) \in H$ since H is a left ideal. But $\alpha\ (y, z)\ [\alpha\ (x, y) + \alpha\ (x, 0)] = \alpha\ (y, z)\ \alpha\ (x, y)$ $= \alpha\ (x, z)$ and $\alpha\ (y, z)\ \alpha\ (x, 0) = \alpha\ (y, z)\ (0) = 0$ since $\alpha\ (y, z) \in T_0(G)$. Hence $\alpha\ (x, z) \in H$, from which it follows that $H = P_x$. Therefore P_x is a minimal left ideal.

By Proposition 3.6, P_x is a $T_0(G)$-subgroup. To show P_x is minimal, let U be any $T_0(G)$-subgroup contained in P_x. Choose $0 \neq \alpha\ (x, y) \in U$ and $\alpha\ (x, z) \in P_x$. Then $\alpha\ (y,z)\ \alpha\ (x, y) \in U$ so $\alpha\ (x, z) \in U$. Therefore $P_x \subseteq U$ so $U = P_x$. Thus P_x is a minimal $T_0(G)$-subgroup.

For any $\alpha\ (x, y) \in P_x$, $\alpha\ (x, y)\ \alpha\ (x, x) = \alpha\ (x, y)$ so $\alpha\ (x, x)$ acts as a right identity for P_x. Also $\alpha\ (x, y) = \alpha\ (x, y)\ \alpha\ (x, x) \in T_0 \circ \alpha\ (x, x)$, and for $\beta \circ \alpha\ (x, x) \in T_0 \circ \alpha\ (x, x)$, $\beta \circ \alpha\ (x, x) = \alpha\ (x, \beta\ (x)) \in P_x$. Thus $P_x = T_0 \circ \alpha\ (x, x)$, so P_x is generated by $\alpha\ (x, x)$.

Proposition 3.8. The group $(P_x, +)$ is isomorphic to G.

Proof: Define f from G to P_x by $f(g) = \alpha\ (x, g)$. Then f is a group homomorphism, since for $g, h \in G$ $f(g + h) = \alpha\ (x, g + h) = \alpha\ (x, g) + \alpha\ (x, h) = f(g) + f(h)$. Also $\alpha\ (x, g) = \alpha\ (x, h)$ implies $\alpha\ (x, g)\ (t) = \alpha\ (x, h)\ (t)$ for $t \in G$. If $x = t$, $\alpha\ (x, g)\ (t) = g$ and $\alpha\ (x, h)\ (t) = h$. Thus $g = h$

which implies f is one-to-one. Clearly f is onto, from which it follows that f is a group isomorphism from P_x to G.

Note 3.9. If $x \in G \sim \{0\}$ and $V = G \sim \{0, x\}$, then $P_x \cap \Sigma \{P_y \mid y \in V\} = \{0\}$. Therefore $P = \Sigma_G P_x$ is a (group) direct sum.

Lemma 3.10. If F is a family of left ideals from a near-ring N, then $\Sigma \{M_i \mid M_i \in F\}$ is a left ideal of N.

Proof: Suppose F is a family $\{M_i\}_I$ of left ideals from a near-ring N. Then it is well-known that $\Sigma_I M_i$ is a normal subgroup of N. Let $m = \sum_{i=1}^{k} m_i$ be any element in $\Sigma_I M_i$, where $m_i \in M_i$. To show that $\Sigma_I M_i$ is a left ideal of N, we must show that for $n_1, n_2 \in N$, $n_2 (m + n_1) - n_2 n_1 \in \Sigma_I M_i$. This shall be shown by induction on k. If $k = 1$, then $n_2 (m + n_1) - n_2 n_1 =$

$n_2 (m_1 + n_1) - n_2 n_1 \in M_1 \subseteq \Sigma_I M_i$ since M_1 is a left ideal.

Let $c_j = \sum_{i=1}^{j} m_i$. Then $n_2 (m_k + (c_{k-1} + n_1)) - n_2 (c_{k-1} + n_1) \in \Sigma_I M_i$ since $c_{k-1} + n_1 \in N$. By hypothesis, $n_2 (c_{k-1} + n_1) - n_2 n_1 \in \Sigma_I M_i$. Hence $n_2 (m + n_1) - n_2 n_1$

$= n_2 [(m_k + \Sigma m_i) + n_1] - n_2 n_1 = n_2 [(m_k + c_{k-1}) + n_1)] - n_2 n_1$

$= \{n_2 [m_k + (c_{k-1} + n_1)] - n_2 (c_{k-1} + n_1)\} + \{n_2 (c_{k-1} + n_1) - n_2 n_1\}$

$\in \Sigma_I M_i$. Therefore $\Sigma_I M_i$ is a left ideal of N.

Theorem 3.11. P is a left ideal of $T_0(G)$.

Proof: By Remark 3.2, P is a (direct) sum of left ideals, so as a direct consequence of Lemma 3.10, P is a left ideal of $T_0(G)$.

Theorem 3.12. If G is finite, then $T_0(G) = P = \oplus \Sigma P_x$ is a direct sum of minimal left ideals.

Proof: Since G is finite, $|G| = n$ where n is some positive integer. By Proposition 3.8, P_x is isomorphic as a group to G, so $|P_x| = n$. Hence $P = \oplus \Sigma P_x$ has order n^{n-1} since $x \neq 0$. By Theorem 3.11, $P \subseteq T_0(G)$. But $|T_0(G)| = n^{n-1}$ since for $\alpha \in T_0(G)$, $\alpha(0) = 0$. Hence $T_0(G) = P = \oplus \Sigma P_x$.

One should note that Theorem 3.12 does not hold if G is infinite, since in this case $|T_0(G)| = |G|^{|G|}$, while $|P| = |G|$.

SUMMARY

In conclusion, we have examined the basic properties of near-rings, and subsequently, we have developed fundamental theorems based on these properties. Among the theorems proved were the Factor Theorem, the Isomorphism Theorems, the Correspondence Theorem, and several theorems concerning near-ring embeddings. One of these results is the fact that if the near-ring N is properly contained in the subgroup G, then N can be embedded in the near-ring with identity, $T(G)$. At this point, one might conjecture that N must be contained in near-rings with identities that are 'smaller' than $T(G)$. Also it would seem that there are better and more interesting ways of embedding a near-ring in a near-ring with identity. However, the author was unable to resolve these questions.

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